On conformal capacity and Teichmüller's modulus problem in space

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Abstract: We solve an extremal problem for the conformal capacity of certain space condensers. The extremal condenser is conformally equivalent to Teichmüller's ring. As an application, we give a dimension-free estimate for the minimal conformal capacity of the condensers with plates E, F such that $a, b \in E$, $c, d \in F$, where a, b, c, d are given points in $\overline{\mathbb{R}}^n$.

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1. Introduction and results

Notation: Let $\mathbb{R}^n$, $n \geq 2$, be the n -dimensional euclidean space, and $\overline{\mathbb{R}^n}$ its one-point compactification $\mathbb{R}^n \cup \{\infty\}$. The norm of a vector $x \in \mathbb{R}^n$ is written as $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$, where $x_1, \dots, x_n$ are the coordinates of x . By $e_1, e_2, \dots, e_n$ we denote the standard unit vectors of $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we set $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$, and for $x \neq O$, $[x, \infty] = \{tx : t \geq 1\} \cup \{\infty\}$, where O is the origin. The $(n-1)$ -dimensional sphere in $\mathbb{R}^n$ with center a and radius r will be denoted by $S^{n-1}(a, r)$.

A *condenser* in $\overline{\mathbb{R}^n}$ is a pair (A, B) , where A, B are nonempty, disjoint, compact subsets of $\overline{\mathbb{R}^n}$ with $\infty \in A$. The complementary sets A and B are the *plates* of the condenser, and the open set $R = \mathbb{R}^n \setminus (A \cup B)$ is the *field* of the condenser. If both A and B are connected, then the condenser (A, B) is called a *ring*. The *conformal capacity* of (A, B) is defined as

$$(1.1) \quad \text{cap}(A, B) = \inf_u \int_R |\nabla u|^n dx$$

where the infimum is taken over all nonnegative $C^\infty(R \cup B)$ functions with compact support in $R \cup B$ such that $u(x) \geq 1$, for $x \in B$ (cf. [AVV] p.165).

Inequalities for the conformal capacity of condensers play an important role in quasiconformal maps (see, e.g. [G1], [V1], [AVV]). In this paper we study rings whose plates are rectilinear segments or half-lines. For such rings F.W.Gehring (see [V1] Lemma 5.27) proved the following result which has many applications in the distortion theory of quasiconformal mappings (see [V1], [AVV]).

1.2. Theorem. *Let $e \in S^{n-1}(O, 1)$ and $s > 1$. Then*

$$(1.3) \quad \text{cap}([O, e_1], [se_1, \infty]) \geq \text{cap}([O, e], [se_1, \infty]).$$

Figure 1: An illustration for Theorem 1.2 .

We shall prove the following stronger result.

1.4. Theorem. *Let $0 < r \leq s \leq t \leq \rho$, $\theta \in [0, \pi/2]$ and $e = (\cos \theta) e_1 + (\sin \theta) e_n$. Consider the sets $E = [-re, se]$, $E^\# = [-re_1, se_1]$, $F = \{le_1 : l \geq t, l \leq -\rho\}$. Then*

$$(1.5) \quad \text{cap}(E, F) \leq \text{cap}(E^\#, F).$$

Figure 2: An illustration for Theorem 1.4 .

The inequality (1.5) implies the “conjecture for chapter 8” in the Appendix F of the book [AVV] by G.D.Anderson, M.K.Vamanamurthy, and M.Vuorinen.

For $n = 2$, an even stronger result can be easily proved: *Let $f(\theta)$ denote the conformal capacity of the condenser (E, F) . Then f is strictly decreasing for $\theta \in (0, \pi/2)$.* This can be proved by mapping the domain $\mathbb{R}^2 \setminus E^\#$ conformally onto $\{z : |z| > 1\}$ and then using simple geometric arguments involving ellipses. Such a monotonicity result probably holds in every dimension.

The proof of Theorem 1.4 is given in Section 3.

In Section 4 we apply Theorem 1.4 to Teichmüller’s modulus problem in space. For $x \in \mathbb{R}^n \setminus \{O, e_1\}$, let

$$(1.6) \quad p(x) = \inf_{E, F} \text{cap}(E, F),$$

where the infimum is taken over all pairs of continua E and F with $O, e_1 \in E$ and $x, \infty \in F$. For $n = 2$, the problem of determining the value of $p(x)$ for all x and finding the corresponding extremal continua was posed by O.Teichmüller [Te]. For further work in the two-dimensional case we refer to [Ku] ch.5 and [SV], and to the references therein. For $n \geq 3$, the problem was studied in [V1] ch.8, [V2] and [AVV] ch.15.

Here we are interested in upper bounds of the form

$$p(x) \leq c \tau_n(\min\{|x|, |x - e_1|\}),$$

where c is a constant and $\tau_n(r)$ is the conformal capacity of Teichmüller’s ring $([-e_1, O], [re_1, \infty])$. By symmetry, we may assume that $|x - e_1| \leq |x|$. Vuorinen [V2] proved that

$$p(x) \leq \tau_n \left(\frac{|x| + |x - e_1| - 1}{2} \right), \quad x \in \mathbb{R}^n \setminus [O, e_1].$$

This implies ([AVV] p.304) that

$$(1.7) \quad p(x) \leq \sqrt{2} \tau_n(|x - e_1|), \quad |x| > 1.$$

For general $x \in \mathbb{R}^n \setminus \{e_1\}$, $|x - e_1| \leq |x|$, [AVV] gives the bound

$$(1.8) \quad p(x) \leq 2^n \sqrt{3} \tau_n(|x - e_1|).$$

Vuorinen posed the problem whether the number $2^n\sqrt{3}$ in (1.8) can be replaced by a number independent of n , (see [V2] 3.11 open problem (2) and [AVV] p.312). We answer this question in the affirmative:

1.9. Theorem. *For $x \in \mathbb{R}^n \setminus \{e_1\}$, $|x - e_1| \leq |x|$,*

$$(1.10) \quad p(x) \leq 4 \tau_n(|x - e_1|).$$

The proof of Theorem 1.9 is based on Theorem 1.4 and on V.N.Dubinin's polarization inequalities for condensers [Du]. In Section 2 we review some known results on conformal capacity and polarization that are necessary for the proofs of the theorems.

2. Background

2.1. Conformal capacity and extremal length

Let Γ be a family of curves in $\overline{\mathbb{R}}^n$. By $\mathcal{F}_\Gamma$ we denote the family of all nonnegative Borel functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$(2.2) \quad \int_{\gamma} \rho ds \geq 1,$$

for every locally rectifiable curve $\gamma \in \Gamma$. The *modulus* $M(\Gamma)$ and the *extremal length* $\lambda(\Gamma)$ are defined by

$$(2.3) \quad M(\Gamma) = \inf_{\rho} \int_{\mathbb{R}^n} \rho^n dm,$$

$$(2.4) \quad \lambda(\Gamma) = M(\Gamma)^{1/(1-n)},$$

where the infimum is taken over all $\rho \in \mathcal{F}_\Gamma$ and m is the n -dimensional Lebesgue measure.

If $E, F \subset \overline{\mathbb{R}}^n$ and Γ is the family of all curves joining E and F then we write $M(E, F) = M(\Gamma)$ and $\lambda(E, F) = \lambda(\Gamma)$. For the properties of modulus and extremal length we refer to [AVV] ch.8 and the references therein. Here we mention only two main properties:

2.5. Theorem. *Let f map the domain $D \subset \overline{\mathbb{R}}^n$ conformally onto a domain $D' \subset \overline{\mathbb{R}}^n$. Then $M(f(\Gamma)) = M(\Gamma)$, for each curve family Γ in D' .*

A theorem due to F.W.Gehring [G2] and W.P.Ziemer [Zi] states that the conformal capacity of a condenser is equal to the modulus of the family of curves joining its plates:

2.6. Theorem. *If (E, F) is a condenser in $\overline{\mathbb{R}}^n$, then*

$$(2.7) \quad \text{cap}(E, F) = M(E, F).$$

2.8. Teichmüller's ring

The plates of Teichmüller's ring are $[-e_1, O]$ and $[re_1, \infty]$, where $r > 0$. Its capacity is denoted by $\tau_n(r)$. For $n = 2$, τ_n has an explicit expression in terms of elliptic integrals (see [AVV] ch.8). For $n \geq 3$, there are no known expressions and so we rely on qualitative properties and inequalities: τ_n is a strictly decreasing function of r and satisfies the following inequality ([AVV] Theorem 11.25)

$$(2.9) \quad \tau_n(cr) \leq \frac{1}{\sqrt{c}} \tau_n(r), \quad 0 < c \leq 1, \quad r > 0.$$

The ring $([-e_1, e_1], [-be_1, \infty] \cup [be_1, \infty])$ can be mapped by a Möbius transformation onto Teichmüller's ring. Using this fact we easily find that (see [AVV] exercise 7.52(c))

$$(2.10) \quad \text{cap}([-e_1, e_1], [-be_1, \infty] \cup [be_1, \infty]) = \tau_n(t),$$

where $t = (b - 1)^2 / (4b)$.

2.11. Polarization

Polarization is a geometric transformation in $\mathbb{R}^n$ discovered by V.Wolontis [Wo]. The behaviour of the conformal capacity of condensers under polarization was investigated by V.N.Dubinin [Du] who used it to solve problem 7.57 in [BBH].

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $x^* = (x_1, \dots, x_{n-1}, -x_n)$. Let also $\infty^* = \infty$. If A is a compact subset of $\mathbb{R}^n$, let $A^* = \{x^* : x \in A\}$, $A^+ = \{x \in A : x_n \geq 0\}$, $A^- = \{x \in A : x_n \leq 0\}$. The *polarization* $\mathbb{P}A$ of A with respect to the oriented $(n-1)$ -dimensional plane $\{x_n = 0\}$ is defined by

$$(2.12) \quad \mathbb{P}A = (A \cup A^*)^+ \cup (A \cap A^*)^-.$$

If (A, B) is a condenser in $\mathbb{R}^n$ the polarization $\mathbb{P}(A, B)$ of (A, B) with respect to $\{x_n = 0\}$ is the condenser

$$(2.13) \quad \mathbb{P}(A, B) = (\mathbb{P}A, (\mathbb{P}B)^*).$$

Note that $\mathbb{P}(A, B) = \mathbb{P}(B^*, A^*)$ and therefore

$$\text{cap } \mathbb{P}(A, B) = \text{cap } \mathbb{P}(B, A).$$

Figure 3: A condenser (A, B) in $\mathbb{R}^2$ and its polarization $(\mathbb{P}A, (\mathbb{P}B)^*)$.

Now, let H be an oriented $(n-1)$ -dimensional plane or sphere in $\mathbb{R}^n$. Let T be a Mbius transformation that maps $\{x_n = 0\}$ (with its natural orientation) onto H preserving their orientation. Then we define the polarization $\mathbb{P}_H A$ of a compact set $A \subset \mathbb{R}^n$ by

$$(2.14) \quad \mathbb{P}_H A = T \mathbb{P} T^{-1} A$$

and similarly we define the polarization $\mathbb{P}_H(A, B)$ of the condenser (A, B) with respect to H .

Dubinin [Du] proved the following theorem:

2.15. Theorem. *If (A, B) is a condenser in $\mathbb{R}^n$ and H is an oriented $(n-1)$ -dimensional plane or sphere, then*

$$(2.16) \quad \text{cap}(A, B) \geq \text{cap}_{\mathbb{P}_H}(A, B).$$

2.17. Remark. Theorem 1.2 follows easily from Theorem 2.15.

3. The inequality for conformal capacity

In this section we prove Theorem 1.4. We start with three lemmas.

3.1. Lemma. *Let k, l, ϵ be real numbers such that $k < l$, $k + l > 0$ and $l + \epsilon < 1$. Then*

$$(3.2) \quad \lambda([ke_1, le_1], S^{n-1}(O, 1)) \geq \lambda[(k + \epsilon)e_1, (l + \epsilon)e_1], S^{n-1}(O, 1)).$$

Proof. The inequality (3.2) follows at once by applying a polarization with respect to the $(n-1)$ -dimensional plane which is perpendicular to the e_1 -axis at the point $(k+l+\epsilon)e_1/2$, and has orientation such that $[(k+\epsilon)e_1, (l+\epsilon)e_1]$ is the polarization of $[ke_1, le_1]$.

3.3. Lemma. *Let $b \leq 0 \leq k \leq l$ with $l - b < 1$. For $z \in \mathbb{C}$, define*

$$g(z) = \frac{z + b}{1 + bz}.$$

Then

$$(3.4) \quad g(-b - l) \leq g(-b - k) \leq g(-b + k) \leq g(-b + l),$$

$$(3.5) \quad g(-b - k) - g(-b - l) \leq g(-b + l) - g(-b + k),$$

$$(3.6) \quad g(-b) - g(-b - k) \leq g(-b + k) - g(-b),$$

$$(3.7) \quad g(-b) - g(-b - l) \leq g(-b + l) - g(-b).$$

Proof. The function g is a Mbius transformation that maps the unit disk onto itself with $g(0) = b$, $g(-b) = 0$, $g(1) = 1$, $g(-1) = -1$. Hence g preserves the diameter $[-1, 1]$ and its orientation. In particular (3.4) holds. The inequality (3.5) follows from the fact that $g(x)$ is a convex function on $[-1, 1]$ ($b \leq 0 \Rightarrow g'' \geq 0$). Finally (3.6) and (3.7) follow from the fact that g preserves hyperbolic distances (or by a simple computation).

3.8. Lemma. *Let $a \leq 0 < r \leq s < R$, $\theta \in [0, \pi/2]$, $e = (\cos \theta) e_1 + (\sin \theta) e_n$, and $E = [-re, se]$. Let γ be the hyperbolic geodesic segment of $S = S^{n-1}(ae_1, R)$ that lies in the $e_1 e_n$ -plane, is tangent to E at the origin, and has end-points $A \in S^{n-1}(O, r) \cap \{x_n \leq 0\}$ and $B \in S^{n-1}(O, s) \cap \{x_n \geq 0\}$. Then*

$$(3.9) \quad \lambda(E, S) \geq \lambda(\gamma, S).$$

Proof. Let j be the reflection with respect to $S^{n-1}(O, r)$. Then $j(E)$ and $j(\gamma)$ lie on two parallel straight lines l_R and l_γ , respectively. Also, $j(S)$ is a sphere centered at $r^2 e_1/a$ and such that

$$j(S) \subset j(S^{n-1}(O, s)) \subset S^{n-1}(O, r).$$

Moreover, l_γ is perpendicular to $j(S)$, while l_E is perpendicular to $S^{n-1}(O, r)$ and to $j(S^{n-1}(O, s))$. These geometric relations and a polarization $\mathbb{P}_H$ with respect to the $(n-1)$ -dimensional plane H which is perpendicular to the $e_1 e_n$ -plane and such that $\mathbb{P}_H(l_\gamma) = l_R$ show that (3.9) holds.

3.10. Proof of Theorem 1.4

By (2.4) and Theorem 2.6, it suffices to prove

$$(3.11) \quad \lambda(E, F) \geq \lambda(E^\#, F).$$

Let $S = S^{n-1}(ae_1, R)$ be the sphere in $\mathbb{R}^n$ such that F is the reflection of $E^\#$ in S . A direct calculation gives

$$(3.12) \quad a = \frac{ts - r\rho}{s + t + r + \rho},$$

$$(3.13) \quad R = \sqrt{(r+a)(\rho+a)} = \sqrt{(t-a)(s-a)}.$$

Case 1: $ts \leq r\rho$.

In this case $a \leq 0$, because of (3.12). By symmetry and subadditivity of extremal length (see [AVV] Lemmas 8.15 and 8.5(2)),

$$(3.14) \quad \lambda(E^\#, F) = \lambda(E^\#, S) + \lambda(S, F),$$

$$(3.15) \quad \lambda(E, F) \geq \lambda(E, S) + \lambda(S, F).$$

So, it suffices to prove

$$(3.16) \quad \lambda(E, S) \geq \lambda(E^\#, S).$$

We identify the $e_1 e_n$ -plane with $\mathbb{C}$ and the e_1 -axis with $\mathbb{R}$ (just in order to use complex notation). Let Γ be the geodesic of S that lies in $\mathbb{C}$ and is tangent

to E at the origin. Let γ be the subarc of Γ with end-points A, B such that $A \in S^{n-1}(O, r) \cap \{x_n \leq 0\}$ and $B \in S^{n-1}(O, s) \cap \{x_n \geq 0\}$. By Lemma 3.8,

$$(3.17) \quad \lambda(E, S) \geq \lambda(\gamma, S).$$

Consider the conformal mapping F that maps S onto $S_o := S^{n-1}(O, 1)$ with $F(O) = O$, $F(\overline{\mathbb{C}}) = \overline{\mathbb{C}}$, and $F(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. Then the restriction of F to $\mathbb{C}$ is given by

$$F = T \circ T_2 \circ T_1,$$

where $T_1(z) = z - a$ is a translation, $T_2(z) = z/R$ is a stretching, and T is the Möbius map given by

$$T(z) = \frac{z + a/R}{1 + za/R}.$$

Let $\delta := F(\gamma)$. Then, by the conformal invariance of extremal length (Theorem 2.5),

$$(3.18) \quad \lambda(\gamma, S) = \lambda(\delta, S_o).$$

Since F preserves geodesics and δ passes through the center O of S_o , we conclude that δ is a rectilinear segment. The smallest angle between δ and $\mathbb{R}$ is equal to θ . We write $\delta = \delta(\theta)$. Let

$$\hat{s} := F(-s), \quad s' := F(s), \quad \hat{r} := F(-r), \quad r' := F(r).$$

Then, applying Lemma 3.3 (with $b = a/R$) we obtain

$$\hat{s} \leq \hat{r} \leq r' \leq s', \quad \hat{r} - \hat{s} \leq s' - r', \quad -\hat{r} \leq r', \quad -\hat{s} \leq s'.$$

These inequalities show that the centers r_1, s_1 of the circles $F(S^{n-1}(O, r)) \cap \mathbb{C}$ and $F(S^{n-1}(O, s)) \cap \mathbb{C}$, respectively, satisfy the inequalities $0 \leq r_1 \leq s_1$. Therefore $\delta(\theta)$ contains a chord of the circle $F(S^{n-1}(O, r)) \cap \mathbb{C}$ and is contained in a chord of the circle $F(S^{n-1}(O, s)) \cap \mathbb{C}$; and these chords are diameters only if $\theta = 0$.

Let $\delta'(\theta) := e^{-i\theta} \delta(\theta)$. Then $\delta'(\theta)$ is a segment $[\sigma_1, \sigma_2] \subset \mathbb{R}$ centered at a point of the positive real axis and $\sigma_2 \leq s'$. Moreover, for the length $|\delta'(\theta)|$ of $\delta'(\theta)$ we have

$$|\delta'(\theta)| = |\delta(\theta)| \leq |\delta(0)| = |\delta'(0)|.$$

Because of Lemma 3.1 these geometric properties imply

$$(3.19) \quad \lambda(\delta(\theta), S_o) = \lambda(\delta'(\theta), S_o) \geq \lambda(\delta(0), S_o) = \lambda(E^\#, S).$$

The last equality comes from conformal invariance; note that

$$F(E^\#) = F([-r, s]) = [\hat{r}, s'] = \delta(0).$$

Now, (3.17), (3.18), and (3.19) show that (3.16) holds and the theorem is proved in case 1.

Case 2: $ts > r\rho$.

In this case we apply a reflection in $S^{n-1}(O, 1)$. Let $\mathcal{R}$ denote this reflection. By conformal invariance, (3.11) is equivalent to

$$(3.20) \quad \lambda(\mathcal{R}(E), \mathcal{R}(F)) \geq \lambda(\mathcal{R}(E^\#), \mathcal{R}(F)).$$

But

$$\begin{aligned} \mathcal{R}(E^\#) &= \{le_1 : l \leq r^{-1}, l \geq s^{-1}\}, \\ \mathcal{R}(F) &= [-\rho^{-1}e_1, t^{-1}e_1], \\ \mathcal{R}(E) &= \{le : l \geq r^{-1}, l \leq s^{-1}\}. \end{aligned}$$

Define

$$\begin{aligned} \tilde{E} &:= [-\rho^{-1}e, t^{-1}e], \\ \tilde{F} &:= \{le_1 : l \leq -r^{-1}, l \geq s^{-1}\}, \\ \tilde{E}^\# &:= [-\rho^{-1}e_1, {}^{-1}e_1]. \end{aligned}$$

A rotation shows that (3.20) is equivalent to

$$(3.23) \quad \lambda(\tilde{E}, \tilde{F}) \geq \lambda(\tilde{E}^\#, \tilde{F}).$$

Let

$$\tilde{r} := \rho^{-1}, \quad \tilde{s} := t^{-1}, \quad \tilde{t} := s^{-1}, \quad \tilde{\rho} := r^{-1}.$$

Then $\tilde{t}\tilde{s} < \tilde{r}\tilde{\rho}$, and hence (3.23) follows from Case 1. Thus the theorem is now proved.

4. Teichmüller's modulus problem

This section contains the proof of Theorem 1.9. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{O, e_1\}$. To prove the theorem, we may assume that $x_2 = \dots = x_{n-1} = 0$ and $|x - e_1| \leq |x| \leq 1$. Let Ω be the region in the e_1e_n -plane defined by these conditions. So we have to prove that

$$(4.1) \quad p(x) \leq 4\tau_n(|x - e_1|), \quad x \in \Omega.$$

By the definition of $p(x)$, if E, F are curves in $\overline{\mathbb{R}}^n$ with $O, e_1 \in E$ and $x, \infty \in F$ then $p(x) \leq M(E, F)$. Moreover, by the subadditivity of moduli ([AVV] Lemma

8.3(3)), if $F = F_1 \cup F_2$ then $M(E, F) \leq M(E, F_1) + M(E, F_2)$. This type of argument will be repeatedly used in the proof below.

We will prove four lemmas. Theorem 1.9 is an immediate consequence of Lemmas 4.2, 4.13 and 4.16.

4.2. Lemma. *Suppose that $p(x) \leq 4\tau_n(|x - e_1|)$ for all $x \in [e_1/2, e_1] \cup [e_1/2, e_1/2 + \sqrt{3}e_n/2]$. Then*

$$p(x) \leq 4\tau_n(|x - e_1|) \text{ for all } x \in \Omega.$$

Proof. Suppose first that $x \in \Omega$ and $|x - e_1| \leq 1/2$. By applying a polarization with respect to a suitable $(n-1)$ -dimensional plane perpendicular to the e_1e_n -plane and passing from e_1 , we deduce that $p(x) \leq p(x')$, where $x' = (1 - |x - e_1|)e_1$ is the circular projection of x on $[e_1/2, e_1]$. On the other hand, by the definition of x' , $|x - e_1| = |x' - e_1|$.

Suppose next that $x \in \Omega$ and $|x - e_1| > 1/2$. We use the same polarization argument as above, but in this case we circularly project x on $[e_1/2, e_1/2 + \sqrt{3}e_n/2]$.

4.3. Lemma. *Let $E = [-e_1, e_1]$ and $F = [-e_n, \infty]$. Then*

$$(4.4) \quad \text{cap}(E, F) \leq \tau_n(1) + \tau_n(1/3),$$

$$(4.5) \quad \text{cap}(E, F) \leq 2\tau_n(1/2).$$

Proof. Because of Theorem 2.6, we can use the modulus $M(E, F)$ instead of $\text{cap}(E, F)$. Let $F_1 = [-e_n, -3e_n]$, $F_2 = [-3e_n, \infty]$. Then

$$(4.6) \quad M(E, F) \leq M(E, F_1) + M(E, F_2).$$

Now, by Theorem 1.4,

$$(4.7) \quad M(E, F_2) \leq \tau_n\left(\frac{3-1}{2}\right) = \tau_n(1).$$

To estimate $M(E, F_1)$ we apply a polarization with respect to the sphere $S^{n-1}(-3e_n/2, 3/2)$ and get

$$(4.8) \quad M(E, F_1) \leq M(E, [-3e_n, \infty] \cup [3e_n, \infty]).$$

But by Theorem 1.4 and (2.10),

$$(4.9) \quad M(E, [-3e_n, \infty] \cup [3e_n, \infty]) \leq \tau_n\left(\frac{1}{3}\right).$$

Hence (4.4) is proved.

To prove (4.5), we let $L_1 = [-e_n, -2e_n]$ and $L_2 = [-2e_n, \infty]$. Then L_2 is the reflection of L_1 in the sphere $S^{n-1}(-e_n, 1)$. By a polarization with respect to that sphere,

$$(4.10) \quad M(E, L_1) \leq M(E, L_2).$$

Hence

$$(4.11) \quad M(E, F) \leq M(E, L_1) + M(E, L_2) \leq 2M(E, L_2).$$

Finally, by Theorem 1.4,

$$(4.12) \quad M(E, L_2) \leq \tau_n \left(\frac{2-1}{2} \right) = \tau_n \left(\frac{1}{2} \right)$$

and the lemma is proved.

4.13. Lemma. *Let $x = e_1/2 + x_n e_n \in \Omega$. Then*

$$p(x) \leq 3.9 \tau_n(|x - e_1|).$$

Proof. We identify the $e_1 e_n$ -plane with $\mathbb{C}$ and the e_1 -axis with $\mathbb{R}$. We consider the Mbius map T on $\overline{\mathbb{R}^n}$ whose restriction to $\mathbb{C}$ has the form

$$T(z) = \frac{1 - (2z - 1)i}{2z - 1 - i}.$$

Then $T(0) = -1$, $T(1) = 1$, $T(\infty) = -i$, $T(1/2 - i/2) = \infty$, and

$$T(x) = T(e_1/2 + x_n e_n) = \frac{1 + 2x_n i}{1 - 2x_n}.$$

Let $E = [-1, 1]$, $F_1 = [-i, -3i]$, $F_2 = [i, 3i]$, $F_3 = [-3i, \infty] \cup [3i, \infty]$.

Case 1: $0 \leq x_n \leq 1/4$.

In this case $T(x) \in [i, 3i]$. Hence,

$$(4.14) \quad p(x) \leq M(E, F_1) + M(E, F_2) + M(E, F_3).$$

As in the proof of Lemma 4.3,

$$(4.15) \quad M(E, F_1) = M(E, F_2) \leq M(E, F_3) \leq \tau_n(1/3).$$

Therefore, by (4.14) and (2.9),

$$\begin{aligned} p(x) &\leq 3\tau_n \left(\frac{1}{3} \right) = 3\tau_n \left(\frac{1}{3} \frac{4}{\sqrt{5}} \frac{\sqrt{5}}{4} \right) \leq 3\sqrt{\frac{3\sqrt{5}}{4}} \tau_n \left(\frac{\sqrt{5}}{4} \right) \leq \\ &\leq 3.9 \tau_n \left(\frac{\sqrt{5}}{4} \right) \leq 3.9 \tau_n(|x - e_1|). \end{aligned}$$

The last inequality holds because if $x_n \in [0, 1/4]$ then $|x - e_1| = (x_n^2 + (1/2)^2)^{1/2} \leq \sqrt{5}/4$.

Case 2: $1/4 \leq x_n < 1/2$.

In this case $T(x) \in [3i, \infty]$. Hence, by (2.9),

$$\begin{aligned} p(x) &\leq M(E, F_1) + M(E, F_3) \leq \tau_n(1/3) + \tau_n(1/3) \leq 2\sqrt{3}\tau_n(1) \leq \\ &\leq 3.5\tau_n(1) \leq 3.5\tau_n(|x - e_1|). \end{aligned}$$

The last inequality comes from the fact $|x - e_1| \leq 1$, for $x \in \Omega$.

Case 3: $1/2 \leq x_n \leq \sqrt{3}/2$.

In this case $T(x) \in [-i, \infty]$. Hence

$$p(x) \leq M(E, F_1) + M(E, F_3) \leq 3.5\tau_n(|x - e_1|)$$

as in Case 2.

So the lemma is proved in all cases.

4.16. Lemma. *If $x \in [e_1/2, e_1] \setminus \{e_1\}$, $p(x) \leq 4\tau_n(|x - e_1|)$.*

Proof. We apply the same map T as in the proof of Lemma 4.13. Then $T(x)$ lies on the unit circle; (we continue to identify the $e_1 e_n$ -plane with $\mathbb{C}$ as in the proof of Lemma 4.13). Let $T(x) = y = y_1 + iy_n$ and $S = S^{n-1}(y, y_n)$. Consider the following sets: $K_1 = [y, (1 + y_n)y]$, $K_2 = [(1 + y_n)y, \infty]$, $K_3 = [-e_n, -3e_n] \cup [-3e_n, \infty]$, and $E = [-e_1, e_1]$. Then

$$(4.17) \quad p(x) \leq \sum_{j=1}^3 M(E, K_j).$$

By Lemma 4.3,

$$(4.18) \quad M(E, K_3) \leq 2\tau_n\left(\frac{1}{2}\right) \leq 2\tau_n(|x - e_1|).$$

Also, since K_2 is the reflection in S of K_1 , a polarization argument shows that

$$(4.19) \quad M(E, K_1) \leq M(E, K_2).$$

To estimate $M(E, K_2)$, we apply Theorem 1.4 to get

$$(4.20) \quad M(E, K_2) \leq \tau_n\left(\frac{y_n}{2}\right).$$

We compute y_n in terms of $x = x_1 e_1$:

$$T(x) = \frac{1 - (2x_1 - 1)i}{2x_1 - 1 - i} = \frac{2x_1 - 1 + 2x_1(1 - x_1)i}{2x_1^2 - 2x_1 + 1}.$$

Therefore

$$(4.21) \quad \tau_n\left(\frac{y_n}{2}\right) = \tau_n\left(\frac{x_1(1 - x_1)}{2x_1^2 - 2x_1 + 1}\right).$$

But $x_1 \geq 2x_1^2 - 2x_1 + 1$ for all $x_1 \in [1/2, 1]$! Hence (4.21) and the monotonicity of τ_n imply

$$(4.22) \quad \tau_n\left(\frac{y_n}{2}\right) \leq \tau_n(1 - x_1) = \tau_n(|x - e_1|).$$

Now, (4.17), (4.18), (4.19), (4.20), and (4.22) yield

$$(4.23) \quad p(x) \leq 2\tau_n(|x - e_1|) + 2\tau_n(|x - e_1|) = 4\tau_n(|x - e_1|)$$

and the lemma is proved.

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